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CONSTRUCTING INFINITE FAMILIES OF GENERALIZED PYTHAGOREAN TRIPLES

Ronald Skurnick

Department of Mathematics, Computer Science, and Information Technology

Nassau Community College

Garden City, NY 11530

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Abstract: **In this article, we show how to construct an infinite family of generalized Pythagorean triples. Then, we consider how Fermat's Last Theorem relates to this construction.**

Keywords: **Pythagorean triple, primitive Pythagorean triple, convergent infinite geometric series, Fermat's Last Theorem.**

1. INTRODUCTION

If a, b, and c are positive integers, then (a, b, c) is called a Pythagorean triple [2] if $a^2 + b^2 = c^2$. The integers within a Pythagorean triple are usually arranged in increasing order. Some well-known Pythagorean triples are $(3, 4, 5)$, $(8, 15, 17)$, and $(10, 24, 26)$. (a, b, c) is called primitive if $\gcd(a, b, c) = 1$. Note that $(3, 4, 5)$ and $(8, 15, 17)$ are primitive, while $(10, 24, 26)$ is not. We can generalize the concept of a Pythagorean triple as follows. If $a_1, a_2, \ldots, a_{k-1}$, c, n, and k are positive integers, and $k \geq 3$, then we shall call $(a_1, a_2, \ldots, a_{k-1}, c)$ a generalized Pythagorean triple, or an n^{th} degree k-tuple, if ${a_1}^n + {a_2}^n + \ldots + {a_{k-1}}^n = c^n$. We shall arrange the integers within an n^{th} degree k-tuple in non-decreasing order and shall say that $(a_1, a_2, \ldots, a_{k-1}, c)$ is primitive if $\gcd(a_1, a_2, \ldots, a_{k-1}, c) = 1$. For example, $(3, 4, 5, 6)$ is a primitive 3^{rd} degree 4-tuple because $3^3 + 4^3 + 5^3 = 6^3$ and $\gcd(3, 4, 5, 6) = 1$. Recall that an infinite geometric series [1] of the form $1 + x + x^2 + x^3 + \cdots$ converges to $\frac{1}{1-x}$ provided that $|x| < 1$. Fermat's Last Theorem [2], which was proven by English mathematician Andrew Wiles in 1995, states that there exist positive integers a, b, c, and n that satisfy the equation $a^n + b^n = c^n$ if and only if $n = 1$ or 2. In this article, we show how to construct an infinite family of generalized Pythagorean triples from a given n^{th} degree k-tuple. Then, we discuss some consequences of Fermat's Last Theorem that are related to this construction.

2. CONSTRUCTING AN INFINITE FAMILY OF GENERALIZED PYTHAGOREAN TRIPLES

Proposition 1: Given an n^{th} degree k-tuple $(a_1, a_2, \ldots, a_{k-1}, c)$ with $n \geq 1$ and $k \geq 3$, we can construct an n^{th} degree $(k + i(k-2))$-tuple for each $i = 1, 2, \ldots$

Proof: If $(a_1, a_2, \ldots, a_{k-1}, c)$ is an n^{th} degree k-tuple with $n \geq 1$ and $k \geq 3$, then ${a_1}^n + {a_2}^n + \cdots + {a_{k-1}}^n = c^n$. Now, select any one of the integers $a_1, a_2, \ldots$, or a_{k-1} (say, a_1) and form the convergent infinite geometric series whose first term is 1 and whose common ratio is $\left(\frac{a_1}{c}\right)^n$. Then, we have $1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} + \left(\frac{a_1}{c}\right)^{3n} + \cdots = \frac{1}{1 - \left(\frac{a_1}{c}\right)^n} = \frac{1}{\frac{c^n}{c^n} - \frac{{a_1}^n}{c^n}} = \frac{1}{\frac{{a_2}^n + \cdots + {a_{k-1}}^n}{c^n}} = \frac{c^n}{{a_2}^n + \cdots + {a_{k-1}}^n}$. That is, $\frac{c^n}{{a_2}^n + \cdots + {a_{k-1}}^n} = 1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n}\left[1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} + \cdots\right] = 1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} \cdot \frac{c^n}{{a_2}^n + \cdots + {a_{k-1}}^n} = 1 + \left(\frac{a_1}{c}\right)^n + \frac{{a_1}^{2n}}{c^n({a_2}^n + \cdots + {a_{k-1}}^n)}$. Multiplying each side of the equation

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$\frac{c^n}{a_2{}^n + \cdots + a_{k-1}{}^n} = 1 + \left(\frac{a_1}{c}\right)^n + \frac{a_1{}^{2n}}{c^n(a_2{}^n + \cdots + a_{k-1}{}^n)}$ by $c^n(a_2{}^n + \cdots + a_{k-1}{}^n)$ gives us $c^{2n} = c^n(a_2{}^n + \cdots + a_{k-1}{}^n) + a_1{}^n(a_2{}^n + \cdots + a_{k-1}{}^n) + a_1{}^{2n}$. Therefore, $c^{2n} = (a_2c)^n + \cdots + (a_{k-1}c)^n + (a_1a_2)^n + \cdots + (a_1a_{k-1})^n + a_1{}^{2n}$, or equivalently, $(c^2)^n = (a_2c)^n + \cdots + (a_{k-1}c)^n + (a_1a_2)^n + \cdots + (a_1a_{k-1})^n + (a_1{}^2)^n$. Hence, $(a_1{}^2, a_1a_2, \ldots, a_1a_{k-1}, a_2c, \ldots, a_{k-1}c, c^2)$ is an n^{th} degree $(k + (k-2))$-tuple. Also, $\frac{c^n}{a_2{}^n + \cdots + a_{k-1}{}^n} = 1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} + \left(\frac{a_1}{c}\right)^{3n}\left[1 + \left(\frac{a_1}{c}\right)^n + \cdots\right] = 1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} + \left(\frac{a_1}{c}\right)^{3n} \cdot \frac{c^n}{a_2{}^n + \cdots + a_{k-1}{}^n} = 1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} + \frac{a_1{}^{3n}}{c^{2n}(a_2{}^n + \cdots + a_{k-1}{}^n)}$. Multiplying each side of the equation $\frac{c^n}{a_2{}^n + \cdots + a_{k-1}{}^n} = 1 + \left(\frac{a_1}{c}\right)^n + \left(\frac{a_1}{c}\right)^{2n} + \frac{a_1{}^{3n}}{c^{2n}(a_2{}^n + \cdots + a_{k-1}{}^n)}$ by $c^{2n}(a_2{}^n + \cdots + a_{k-1}{}^n)$, we obtain $c^{3n} = c^{2n}(a_2{}^n + \cdots + a_{k-1}{}^n) + a_1{}^nc^n(a_2{}^n + \cdots + a_{k-1}{}^n) + a_1{}^{2n}(a_2{}^n + \cdots + a_{k-1}{}^n) + a_1{}^{3n}$, so $c^{3n} = (a_2c^2)^n + \cdots + (a_{k-1}c^2)^n + (a_1a_2c)^n + \cdots + (a_1a_{k-1}c)^n + (a_1{}^2a_2)^n + \cdots + (a_1{}^2a_{k-1})^n + a_1{}^{3n}$, or equivalently, $(c^3)^n = (a_2c^2)^n + \cdots + (a_{k-1}c^2)^n + (a_1a_2c)^n + \cdots + (a_1a_{k-1}c)^n + (a_1{}^2a_2)^n + \cdots + (a_1{}^2a_{k-1})^n + (a_1{}^3)^n$. This means that $(a_1{}^3, a_1{}^2a_2, \ldots, a_1{}^2a_{k-1}, a_1a_2c, \ldots, a_1a_{k-1}c, a_2c^2, \ldots, a_{k-1}c^2, c^3)$ is an n^{th} degree $(k + 2(k-2))$-tuple. Observe that the given n^{th} degree k-tuple $(a_1, a_2, \ldots, a_{k-1}, c)$ may or may not be primitive. If $(a_1, a_2, \ldots, a_{k-1}, c)$ is primitive, then the n^{th} degree $(k + (k-2))$-tuple $(a_1{}^2, a_1a_2, \ldots, a_1a_{k-1}, a_2c, \ldots, a_{k-1}c, c^2)$ is primitive and the n^{th} degree $(k + 2(k-2))$-tuple $(a_1{}^3, a_1{}^2a_2, \ldots, a_1{}^2a_{k-1}, a_1a_2c, \ldots, a_1a_{k-1}c, a_2c^2, \ldots, a_{k-1}c^2, c^3)$ is primitive if and only if $\gcd(a_1, c) = 1$. Also, the integers in $(a_1{}^2, a_1a_2, \ldots, a_1a_{k-1}, a_2c, \ldots, a_{k-1}c, c^2)$ are all different, and the integers in $(a_1{}^3, a_1{}^2a_2, \ldots, a_1{}^2a_{k-1}, a_1a_2c, \ldots, a_1a_{k-1}c, a_2c^2, \ldots, a_{k-1}c^2, c^3)$ are all different, if and only if the integers in $(a_1, a_2, \ldots, a_{k-1}, c)$ are all different. By iterating this procedure, we can construct an n^{th} degree $(k + i(k-2))$-tuple for each $i = 1, 2, \ldots$.

The following numerical example illustrates how to construct a 3^{rd} degree 6-tuple and a 3^{rd} degree 8-tuple from a given 3^{rd} degree 4-tuple.

Example 1: Let $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, and $c = 6$. Then, $(3, 4, 5, 6)$ is a 3^{rd} degree 4-tuple because $3^3 + 4^3 + 5^3 = 6^3$. Now, select any one of the integers 3, 4, or 5 (say, 3) and form the convergent infinite geometric series whose first term is 1 and whose common ratio is $\left(\frac{3}{6}\right)^3$. Then, $1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 + \left(\frac{3}{6}\right)^9 + \cdots = \frac{1}{1 - \left(\frac{3}{6}\right)^3} = \frac{1}{\frac{6^3}{6^3} - \frac{3^3}{6^3}} = \frac{1}{\frac{4^3+5^3}{6^3}} = \frac{6^3}{4^3+5^3}$. That is, $\frac{6^3}{4^3+5^3} = 1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6\left[1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 + \cdots\right] = 1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 \cdot \frac{6^3}{4^3+5^3} = 1 + \left(\frac{3}{6}\right)^3 + \frac{3^6}{6^3(4^3+5^3)}$. Multiplying each side of the equation $\frac{6^3}{4^3+5^3} = 1 + \left(\frac{3}{6}\right)^3 + \frac{3^6}{6^3(4^3+5^3)}$ by $6^3(4^3+5^3)$ gives us $6^6 = 6^3(4^3+5^3) + 3^3(4^3+5^3) + 3^6$, from which it follows that $6^6 = 24^3 + 30^3 + 12^3 + 15^3 + 3^6$, or equivalently, $36^3 = 24^3 + 30^3 + 12^3 + 15^3 + 9^3$. Therefore, $(9, 12, 15, 24, 30, 36)$ is a 3^{rd} degree 6-tuple. Also, $\frac{6^3}{4^3+5^3} = 1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 + \left(\frac{3}{6}\right)^9\left[1 + \left(\frac{3}{6}\right)^3 + \cdots\right] = 1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 + \left(\frac{3}{6}\right)^9 \cdot \frac{6^3}{4^3+5^3} = 1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 + \frac{3^9}{6^6(4^3+5^3)}$. Multiplying each side of the equation $\frac{6^3}{4^3+5^3} = 1 + \left(\frac{3}{6}\right)^3 + \left(\frac{3}{6}\right)^6 + \frac{3^9}{6^6(4^3+5^3)}$ by $6^6(4^3+5^3)$ yields $6^9 = 6^6(4^3+5^3) + 3^36^3(4^3+5^3) + 3^6(4^3+5^3) + 3^9$, from which we get $6^9 = 6^64^3 + 6^65^3 + 3^34^36^3 + 3^35^36^3 + 3^64^3 + 3^65^3 + 3^9$, or equivalently, $216^3 = 144^3 + 180^3 + 72^3 + 90^3 + 36^3 + 45^3 + 27^3$. This means that $(27, 36, 45, 72, 90, 144, 180, 216)$ is a 3^{rd} degree 8-tuple. Observe that the 3^{rd} degree 4-tuple $(3, 4, 5, 6)$ is primitive, but the 3^{rd} degree 6-tuple $(9, 12, 15, 24, 30, 36)$ and the 3^{rd} degree 8-tuple $(27, 36, 45, 72, 90, 144, 180, 216)$ are not primitive because $\gcd(3, 6) > 1$. Also, the integers in $(9, 12, 15, 24, 30, 36)$ are all different, and the integers in $(27, 36, 45, 72, 90, 144, 180, 216)$ are all different, because the integers in $(3, 4, 5, 6)$ are all different.

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3. SOME NOTEWORTHY CONSEQUENCES OF FERMAT'S LAST THEOREM

Corollary 1: Let n be a positive integer. Then, there exists an n^{th} degree k-tuple $(a_1, a_2, \ldots, a_{k-1}, c)$ for each $k = 3, 4, \ldots$ if and only if n equals 1 or 2.

Proof: Fermat's Last Theorem tells us that n^{th} degree 3-tuples exist if and only if $n = 1$ or 2. By Proposition 1, we know that, given an n^{th} degree 3-tuple, we can construct an n^{th} degree $\big(3 + i(3-2)\big)$-tuple for each $i = 1, 2, \ldots$

Example 2: Let $a, b, c,$ and n be positive integers that satisfy the equation $a^n + b^n = c^n$. Then, by Fermat's Last Theorem, (a, b, c) is an n^{th} degree 3-tuple, where $n = 1$ or 2. Now, select either one of the integers a or b (say, a) and form the convergent infinite geometric series whose first term is 1 and whose common ratio is $\left(\frac{a}{c}\right)^n$. Then, $1 + \left(\frac{a}{c}\right)^n + \left(\frac{a}{c}\right)^{2n} + \left(\frac{a}{c}\right)^{3n} + \cdots = \frac{1}{1-\left(\frac{a}{c}\right)^n} = \frac{1}{\frac{c^n}{c^n} - \frac{a^n}{c^n}} = \frac{1}{\frac{b^n}{c^n}} = \left(\frac{c}{b}\right)^n$. Since $\left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a}{c}\right)^{2n}\left[1 + \left(\frac{a}{c}\right)^n + \left(\frac{a}{c}\right)^{2n} + \cdots\right]$, we have $\left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a^2}{c^2}\right)^n \cdot \left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a^2}{bc}\right)^n$. Multiplying each side of the equation $\left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a^2}{bc}\right)^n$ by $(bc)^n$, we find that $(c^2)^n = (bc)^n + (ab)^n + (a^2)^n$. Thus, (a^2, ab, bc, c^2) is an n^{th} degree 4-tuple. Also, since $\left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a}{c}\right)^{2n} + \left(\frac{a}{c}\right)^{3n}\left[1 + \left(\frac{a}{c}\right)^n + \cdots\right]$, we have $\left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a^2}{c^2}\right)^n + \left(\frac{a^3}{c^3}\right)^n \cdot \left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a^2}{c^2}\right)^n + \left(\frac{a^3}{bc^2}\right)^n$. Multiplying each side of the equation $\left(\frac{c}{b}\right)^n = 1 + \left(\frac{a}{c}\right)^n + \left(\frac{a^2}{c^2}\right)^n + \left(\frac{a^3}{bc^2}\right)^n$ by $(bc^2)^n$, we obtain $(c^3)^n = (bc^2)^n + (abc)^n + (a^2b)^n + (a^3)^n$. This means that $(a^3, a^2b, abc, bc^2, c^3)$ is an n^{th} degree 5-tuple.

Now, if we substitute the values $a = 3$, $b = 4$, and $c = 5$ into (a, b, c), (a^2, ab, bc, c^2), and $(a^3, a^2b, abc, bc^2, c^3)$, respectively, we obtain the following 2^{nd} degree k-tuples for $k = 3, 4,$ and 5: $(3, 4, 5)$, $(9, 12, 20, 25)$, and $(27, 36, 60, 100, 125)$. Note that since $(3, 4, 5)$ is primitive and $\gcd(3, 5) = 1$, $(9, 12, 20, 25)$ and $(27, 36, 60, 100, 125)$ are both primitive. Also, the integers in $(9, 12, 20, 25)$ are all different, and the integers in $(27, 36, 60, 100, 125)$ are all different, because the integers in $(3, 4, 5)$ are all different.

Consider the following two observations about certain infinite geometric series.

Proposition 2: Let $a, c,$ and n be positive integers with $a < c$. Then, the sum, S, of the convergent infinite geometric series $1 + \left(\frac{a}{c}\right)^n + \left(\frac{a}{c}\right)^{2n} + \left(\frac{a}{c}\right)^{3n} + \cdots$, each of whose terms can be expressed as the n^{th} power of a rational number, is equal the n^{th} power of a rational number, $\left(\frac{c}{b}\right)^n$, if and only if there exists a positive integer b such that $a^n + b^n = c^n$.

Proof: Since a and c are positive integers and $a < c$, the sum, S, of this infinite geometric series is $\frac{1}{1-\left(\frac{a}{c}\right)^n} = \frac{1}{\frac{c^n}{c^n} - \frac{a^n}{c^n}} = \frac{c^n}{c^n - a^n}$, which equals the n^{th} power of a rational number if and only if $c^n - a^n = b^n$ for some positive integer b. By Fermat's Last Theorem, such a positive integer b exists if and only if $n = 1$ or 2.

Corollary 2: Suppose $a, b, c,$ and n are positive integers that satisfy the equation $a^n + b^n = c^n$. Let S be the sum of the series $1 + \left(\frac{a}{c}\right)^n + \left(\frac{a}{c}\right)^{2n} + \left(\frac{a}{c}\right)^{3n} + \cdots$, and let S' be the sum of the series $1 + \left(\frac{b}{c}\right)^n + \left(\frac{b}{c}\right)^{2n} + \left(\frac{b}{c}\right)^{3n} + \cdots$. Then, $S + S' = S \cdot S' = \left(\frac{c^2}{ab}\right)^n$ if and only if n equals 1 or 2.

Proof: Since $S = \left(\frac{c}{b}\right)^n$ and $S' = \left(\frac{c}{a}\right)^n$, we have $S + S' = \frac{c^n}{b^n} + \frac{c^n}{a^n} = \frac{c^n(a^n + b^n)}{a^n b^n} = \frac{c^{2n}}{a^n b^n}$ and $S \cdot S' = \frac{c^n}{b^n} \cdot \frac{c^n}{a^n} = \frac{c^{2n}}{a^n b^n}$. Therefore, $S + S' = S \cdot S' = \left(\frac{c^2}{ab}\right)^n$. By Fermat's Last Theorem, this occurs if and only if n equals 1 or 2.

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4. EXTENDING THE CONSTRUCTION TO SUMS AND DIFFERENCES

We conclude this article by demonstrating how to construct equations that express the n^{th} power of a positive integer as an alternating sum of n^{th} powers of positive integers for n equals 1 or 2. This construction can be extended, by a technique similar to that displayed in the proof of Proposition 1, to express the n^{th} power of a positive integer in terms of more general sums and differences of n^{th} powers of positive integers for any positive integer n.

Example 3: Let a, b, c, and n be positive integers, with $a < b$, that satisfy the equation $a^n + b^n = c^n$. Then, by Fermat's Last Theorem, (a, b, c) is an n^{th} degree 3-tuple with $n = 1$ or 2. Now, form the convergent infinite geometric series whose first term is 1 and whose common ratio is $-\left(\frac{a}{b}\right)^n$. Then, $1 - \left(\frac{a}{b}\right)^n + \left(\frac{a}{b}\right)^{2n} - \left(\frac{a}{b}\right)^{3n} + \cdots = \frac{1}{1+\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{b^n}{b^n}+\frac{a^n}{b^n}} = \frac{1}{\frac{c^n}{b^n}} = \left(\frac{b}{c}\right)^n$. Since $\left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a}{b}\right)^{2n}\left[1 - \left(\frac{a}{b}\right)^n + \left(\frac{a}{b}\right)^{2n} - \cdots\right]$, we have $\left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a^2}{b^2}\right)^n \cdot \left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a^2}{bc}\right)^n$. Multiplying each side of the equation $\left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a^2}{bc}\right)^n$ by $(bc)^n$, we see that $(b^2)^n = (bc)^n - (ac)^n + (a^2)^n$. Also, since $\left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a}{b}\right)^{2n} - \left(\frac{a}{b}\right)^{3n}\left[1 - \left(\frac{a}{b}\right)^n + \cdots\right]$, we have $\left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a^2}{b^2}\right)^n - \left(\frac{a^3}{b^3}\right)^n \cdot \left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a^2}{b^2}\right)^n - \left(\frac{a^3}{b^2 c}\right)^n$. Multiplying each side of the equation $\left(\frac{b}{c}\right)^n = 1 - \left(\frac{a}{b}\right)^n + \left(\frac{a^2}{b^2}\right)^n - \left(\frac{a^3}{b^2 c}\right)^n$ by $(b^2 c)^n$ yields $(b^3)^n = (b^2 c)^n - (abc)^n + (a^2 c)^n - (a^3)^n$.

Substituting the values $a = 3$, $b = 4$, $c = 5$, and $n = 2$ into $(b^2)^n = (bc)^n - (ac)^n + (a^2)^n$ and $(b^3)^n = (b^2 c)^n - (abc)^n + (a^2 c)^n - (a^3)^n$, respectively, yields the following two equations: $(4^2)^2 = (4 \cdot 5)^2 - (3 \cdot 5)^2 + (3^2)^2$ and $(4^3)^2 = (4^2 \cdot 5)^2 - (3 \cdot 4 \cdot 5)^2 + (3^2 \cdot 5)^2 - (3^3)^2$. That is, $16^2 = 20^2 - 15^2 + 9^2$ and $64^2 = 80^2 - 60^2 + 45^2 - 27^2$.

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